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REDUCING THREE-DIMENSIONAL ELASTICITY PROBLEMS TO TWO-DIMENSIONAL PROBLEMS BY APPROXIMATING STRESSES AND DISPLACEMENTS BY LEGENDRE POLYNOMIALS

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Shell equations are constructed in orthogonal curvilinear coordinates using approximations of stresses and displacements by Legendre polynomials. The order of the constructed system of differential equations is independent of whether stresses and displacements or their combination are specified on the shell surfaces, which provides the correct formulation of the surface conditions in terms of both displacements and stresses. This allows the system of differential equations of laminated shells to be constructed using matching conditions for displacements and stresses on the contact surfaces.

Key words: *shell equations, Legendre polynomials, elastic curvilinear layer.*

Introduction. Three-dimensional elasticity problems are reduced to two-dimensional problems (shell theory) by using kinematic and force hypotheses [1] or by expanding elastic solutions in terms of a certain complete system of functions [2–5]. The kinematic and force hypotheses impose rather rigorous constraints on the stress–strain state; therefore, these hypotheses are usually used to construct shell equations for the case where stresses are specified on the shell surfaces. Solution of contact problems based on these equations often leads to physically meaningless results. Expansion of elastic solutions with respect to a certain complete system of functions provides shell equations in various approximations. In this case, an important question arises: What additional assumptions should be used to construct a particular approximation, i.e., how many terms should be retained in the expansions to construct the required approximation? Since Legendre polynomials form a complete system of functions in the space $L_2[-1,1]$, exactly this system is usually used to construct shell equations.

In the present paper, first-approximation differential equations of laminated elastic shells are constructed using the approaches set out in [3, 4, 6–8].

1. Three-Dimensional Elasticity Equations in Curvilinear Orthogonal Coordinates. We write plane elasticity equations for the region $\omega = \{\alpha_1, \alpha_2, x_3\colon\ \alpha_i \in [l_i^-, l_i^+],\ x_3 \in [-h/2, h/2],\ i = 1,2\}$.

The equilibrium equations are written as

$$\frac{\partial\hat\sigma_{11}}{\partial\alpha_1} + \frac{\partial\hat\sigma_{12}}{\partial\alpha_2} + \frac{\partial\hat\sigma_{13}}{\partial x_3} + \hat\sigma_{21}A_{12} + \hat\sigma_{31}A_1' - \hat\sigma_{22}A_{21} + q_1H_1H_2 = 0 \quad (1 \rightleftarrows 2),$$
$$\frac{\partial\hat\sigma_{31}}{\partial\alpha_1} + \frac{\partial\hat\sigma_{32}}{\partial\alpha_2} + \frac{\partial\hat\sigma_{33}}{\partial x_3} - \hat\sigma_{11}A_1' - \hat\sigma_{22}A_2' + q_3H_1H_2 = 0, \tag{1.1}$$

where

$$\hat\sigma_{11} = H_2\sigma_{11}, \qquad \hat\sigma_{12} = H_1\sigma_{12}, \qquad \hat\sigma_{13} = H_1H_2\sigma_{13}, \qquad \hat\sigma_{31} = H_2\sigma_{31},$$

$$H_1 = A_1\Big(1 + \frac{x_3}{R_1}\Big), \qquad A_1' = \frac{A_1}{R_1}, \qquad A_{12} = \frac{1}{A_2}\,\frac{\partial A_1}{\partial\alpha_2} \quad (1 \leftrightarrows 2), \quad \hat\sigma_{33} = H_1H_2\sigma_{33},$$

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σ_{ij} are the stresses, q_i are the body forces, and A_1 and A_2 are the Lamé coefficients and R_1 and R_2 are the radii of the principal curvatures of the surface $x_3 = 0$. Here and below, the notation $1 \rightleftarrows 2$ indicates the existence of equations obtained from the previous equations by replacing the subscript 1 by 2 and 2 by 1.

In the linear theory of elasticity, the strain-tensor components e_{ij} are related to the displacement-vector components $\boldsymbol{U}(u_1, u_2, u_3)$ by the formulas

$$e_{11} = \frac{1}{H_1}\Big(\frac{\partial u_1}{\partial \alpha_1} + u_2 A_{12} + u_3 A_1'\Big),$$

$$2e_{12} = \frac{1}{H_1}\Big(\frac{\partial u_2}{\partial \alpha_1} - u_1 A_{12}\Big) + \frac{1}{H_2}\Big(\frac{\partial u_1}{\partial \alpha_2} - u_2 A_{21}\Big), \tag{1.2}$$

$$2e_{31} = 2e_{13} = \frac{1}{H_1}\Big(\frac{\partial u_3}{\partial \alpha_1} - u_1 A_1'\Big) + \frac{\partial u_1}{\partial x_3} \quad (1 \leftrightarrows 2), \qquad e_{33} = \frac{\partial u_3}{\partial x_3}.$$

The relationship between the stress-tensor and strain-tensor components is given by

$$\sigma_{ij} = a_{ijmn} e_{mn}, \qquad i, j = 1, 2, 3. \tag{1.3}$$

In relations (1.3), the coefficients a_{ijmn} satisfy the conditions

$$a_{ijmn}\varepsilon_{ij}\varepsilon_{mn} - c\,\varepsilon_{ij}\varepsilon_{ij} \geq 0, \qquad a_{ijmn} = a_{jimn} = a_{ijnm},$$

where c is a non-negative constant and the summation is performed over the dummy indices from 1 to 3.

The boundary conditions for stresses and displacements are assumed to have the form

$$\Big[a_{im}^{\pm} u_i + (1 - a_{im}^{\pm})\hat{\sigma}_{im}\Big]_{\alpha_m = x_m^{\pm}} = \varphi_{im}^{\pm}, \qquad m = 1, 2,$$

$$\Big[a_{i3}^{\pm} u_i + (1 - a_{i3}^{\pm})\hat{\sigma}_{i3}\Big]_{x_3 = x_3^{\pm}} = \varphi_{i3}^{\pm}, \qquad i = 1, 2, 3, \tag{1.4}$$

where $a_{i3}^{\pm}$ are the specified piecewise constant functions of the variables α_1 and α_2, equal to zero or unity, $\varphi_{i3}^{\pm}$ are the specified piecewise continuous functions of the variables α_1 and α_2, and $a_{im}^{\pm}$ are constants equal to zero or unity.

We note that for an element whose dimension in the x_3-direction is $h = x_3^+ - x_3^-$ and whose dimensions in the α_1 and α_2 directions are infinitely small, the equilibrium equations can be written as

$$\int_{-1}^{1} \Big(\frac{\partial \hat{\sigma}_{11}}{\partial \alpha_1} + \frac{\partial \hat{\sigma}_{12}}{\partial \alpha_2} + \frac{\partial \hat{\sigma}_{13}}{\partial x_3} + \hat{\sigma}_{21} A_{12} + \hat{\sigma}_{31} A_1' - \hat{\sigma}_{22} A_{21} + q_1 H_1 H_2\Big) P_k \, d\zeta = 0$$

$$(1 \leftrightarrows 2), \qquad k = 0, 1, \tag{1.5}$$

$$\int_{-1}^{1} \Big(\frac{\partial \hat{\sigma}_{31}}{\partial \alpha_1} + \frac{\partial \hat{\sigma}_{32}}{\partial \alpha_2} + \frac{\partial \hat{\sigma}_{33}}{\partial x_3} - \hat{\sigma}_{11} A_1' - \hat{\sigma}_{22} A_2' + q_3 H_1 H_2\Big) \, d\zeta = 0,$$

where $P_k(\zeta)$ are Legendre polynomials and $\zeta = (2/h)(x_3 - (x_3^+ + x_3^-)/2)$.

2. Reducing the Three-Dimensional Problem to a Two-Dimensional Problem. To reduce the three-dimensional elasticity problem to a two-dimensional problem, one replaces Eqs. (1.1) by the system of equations

$$\int_{-1}^{1} \Big(\frac{\partial \hat{\sigma}_{11}}{\partial \alpha_1} + \frac{\partial \hat{\sigma}_{12}}{\partial \alpha_2} + \frac{\partial \hat{\sigma}_{13}}{\partial x_3} + \hat{\sigma}_{21} A_{12} + \hat{\sigma}_{31} A_1' - \hat{\sigma}_{22} A_{21} + q_1 H_1 H_2\Big) P_k \, d\zeta \quad (1 \leftrightarrows 2), \quad k = 0, 1, \dots, N; \tag{2.1}$$

$$\int_{-1}^{1} \Big(\frac{\partial \hat{\sigma}_{31}}{\partial \alpha_1} + \frac{\partial \hat{\sigma}_{32}}{\partial \alpha_2} + \frac{\partial \hat{\sigma}_{33}}{\partial x_3} - \hat{\sigma}_{11} A_1' - \hat{\sigma}_{22} A_2' + q_3 H_1 H_2\Big) P_k \, d\zeta = 0, \quad k = 0, 1, \dots, N - 1. \tag{2.2}$$

To construct the shell equations in the Nth approximation, one should approximate the quantities $\hat{\sigma}_{ij}$ in (2.1) and (2.2) by Legendre polynomials (for some quantities, two approximations are used) so that all terms in the integrands are expansions in Legendre polynomials of the same degree:

$$\hat{\sigma}'_{11} = \sum_{k=0}^{k=N} \hat{\sigma}^k_{11} P_k, \qquad \hat{\sigma}'_{12} = \sum_{k=0}^{k=N} \hat{\sigma}^k_{12} P_k, \qquad \hat{\sigma}'_{13} = \sum_{k=0}^{k=N+1} \hat{\sigma}^k_{13} P_k,$$

$$\hat{\sigma}'_{31} = \sum_{k=0}^{k=N} \hat{\sigma}^k_{31} P_k, \qquad \hat{\sigma}'_{33} = \sum_{k=0}^{k=N} \hat{\sigma}^k_{33} P_k, \tag{2.3}$$

$$\hat{\sigma}''_{31} = \sum_{k=0}^{k=N-1} \hat{\sigma}^k_{31} P_k, \qquad \hat{\sigma}''_{11} = \sum_{k=0}^{k=N-1} \hat{\sigma}^k_{11} P_k \qquad (1 \leftrightarrows 2).$$

Here

$$\hat{\sigma}^k_{11} = \frac{1}{2}(1+2k) \int_{-1}^{1} H_2 \sigma_{11} P_k \, d\zeta, \qquad \hat{\sigma}^k_{12} = \frac{1}{2}(1+2k) \int_{-1}^{1} H_1 \sigma_{12} P_k \, d\zeta,$$

$$\hat{\sigma}^k_{13} = \frac{1}{2}(1+2k) \int_{-1}^{1} H_1 H_2 \sigma_{13} P_k d\zeta, \qquad \hat{\sigma}^k_{31} = \frac{1}{2}(1+2k) \int_{-1}^{1} H_2 \sigma_{31} P_k \, d\zeta, \tag{2.4}$$

$$\hat{\sigma}^k_{33} = \frac{1}{2}(1+2k) \int_{-1}^{1} H_1 H_2 \sigma_{33} P_k \, d\zeta \qquad (1 \leftrightarrows 2), \quad k = 0, 1, \ldots, N+1.$$

In view of (2.3) and (2.4), Eqs. (2.1) and (2.2) become

$$\frac{\partial \hat{\sigma}'_{11}}{\partial \alpha_1} + \frac{\partial \hat{\sigma}'_{12}}{\partial \alpha_2} + \frac{\partial \hat{\sigma}'_{13}}{\partial x_3} + \hat{\sigma}'_{21} A_{12} + \hat{\sigma}'_{31} A'_1 - \hat{\sigma}'_{22} A_{21} + \hat{q}_1 = 0 \quad (1 \leftrightarrows 2); \tag{2.5}$$

$$\frac{\partial \hat{\sigma}''_{31}}{\partial \alpha_1} + \frac{\partial \hat{\sigma}''_{32}}{\partial \alpha_2} + \frac{\partial \hat{\sigma}''_{33}}{\partial x_3} - \hat{\sigma}''_{11} A'_1 - \hat{\sigma}''_{22} A'_2 + \hat{q}_3 = 0, \tag{2.6}$$

where

$$\hat{q}_1 = \sum_{k=0}^{k=N} \hat{q}^k_1 P_k \quad (1 \leftrightarrows 2), \qquad \hat{q}_3 = \sum_{k=0}^{k=N-1} \hat{q}^k_3 P_k,$$

$$\hat{q}^k_i = \frac{1}{2}(1+2k) \int_{-1}^{1} q_i H_1 H_2 P_k \, d\zeta, \qquad i = 1, 2, 3.$$

The displacement-vector components u_i are approximated by Legendre polynomials u'_i in such a manner that the polynomials $\hat{\sigma}'_{11}$, $\hat{\sigma}'_{12}$, $\hat{\sigma}'_{13}$, $\hat{\sigma}''_{31}$ $(1 \leftrightarrows 2)$, and $\hat{\sigma}'_{33}$ have the same degree as the polynomials $\partial u'_1/\partial \alpha_1$, $\partial u'_1/\partial \alpha_2$, $\partial u''_1/\partial x_3$, $\partial u'_3/\partial \alpha_1$ $(1 \leftrightarrows 2)$, and $\partial u''_3/\partial x_3$, respectively. Therefore, the displacements are approximated as follows:

$$u'_1 = \sum_{k=0}^{k=N} u^k_1 P_k, \qquad u''_1 = \sum_{k=0}^{k=N+2} u^k_1 P_k \qquad (1 \leftrightarrows 2),$$

$$u'_3 = \sum_{k=0}^{k=N-1} u^k_3 P_k, \qquad u''_3 = \sum_{k=0}^{k=N+1} u^k_3 P_k. \tag{2.7}$$

Multiplying Eqs. (2.5) by u'_i and Eqs. (2.6) by u'_3, summing the resulting equations, and integrating over the region ω, we obtain

$$\int\limits_{\omega} \Big(\frac{\partial \hat\sigma'_{\gamma\delta}}{\partial \alpha_\delta}\, u'_\gamma + \frac{\partial \hat\sigma'_{\gamma 3}}{\partial x_3}\, u'_\gamma + (\hat\sigma'_{21}A_{12} + \hat\sigma'_{31}A'_1 - \hat\sigma'_{22}A_{21})u'_1 + (\hat\sigma'_{12}A_{21} + \hat\sigma'_{32}A'_2 - \hat\sigma'_{11}A_{12})u'_2$$

$$+ \frac{\partial \hat\sigma'_{33}}{\partial x_3}\, u'_3 + \frac{\partial \hat\sigma''_{3\gamma}}{\partial \alpha_\gamma}\, u'_3 - (\hat\sigma''_{11}A'_1 + \hat\sigma''_{22}A'_2)u'_3 + \hat q_i u'_i\Big) d\omega = 0. \tag{2.8}$$

Since $\partial\hat\sigma'_{\gamma 3}/\partial x_3$ $(\gamma = 1, 2)$ are expansions in Legendre polynomials of degree up to Nth, the coefficients u'_γ in the integrand can be replaced by u''_γ, by virtue of the orthogonality of Legendre polynomials. The quantity $\partial\hat\sigma'_{33}/\partial x_3$ is the expansion in Legendre polynomials of degree up to the $N-1$; therefore, the coefficient u'_3 can be replaced by u''_3. In view of the aforesaid, relation (2.8) is written as

$$\int\limits_{\omega} \Big(\hat\sigma'_{\delta\gamma}\, \frac{\partial u'_\delta}{\partial \alpha_\gamma} + \hat\sigma''_{3\gamma}\, \frac{\partial u'_3}{\partial \alpha_\gamma} + \hat\sigma'_{i3}\, \frac{\partial u''_i}{\partial x_3} - (\hat\sigma'_{21}A_{12} + \hat\sigma'_{31}A'_1 - \hat\sigma'_{22}A_{21})u'_1$$

$$- (\hat\sigma'_{12}A_{21} + \hat\sigma'_{32}A'_2 - \hat\sigma'_{11}A_{12})u'_2 + (\hat\sigma''_{11}A'_1 + \hat\sigma''_{22}A'_2)\Big)\, d\omega$$

$$= \int\limits_{\omega} \Big(\frac{\partial}{\partial \alpha_\delta}\, (\hat\sigma'_{\gamma\delta}u'_\gamma + \hat\sigma''_{3\delta}u'_3) + \frac{\partial}{\partial x_3}\, (\hat\sigma'_{i3}u''_i) + \hat q_i u'_i\Big) d\omega. \tag{2.9}$$

In the integrand on the left side of equality (2.9), which we denote by E, the expansions $\hat\sigma'_{ij}$ and $\hat\sigma''_{ij}$ can, by virtue of the orthogonality of Legendre polynomials, be replaced by the expansions

$$\hat\sigma_{ij} = \sum_{k=0}^{k=\infty} \sigma^k_{ij} P_k(\zeta).$$

Then,

$$E = \int\limits_{\omega} \Big(\hat\sigma_{11}\Big(\frac{\partial u'_1}{\partial x_1} + A_{12}u'_2 + A'_1 u'{}_3\Big) + \hat\sigma_{22}\Big(\frac{\partial u'_2}{\partial x_2} + A_{21}u'_1 + A'_2 u'_3\Big)$$

$$+ \hat\sigma_{12}\Big(\frac{\partial u'_1}{\partial x_2} - A_{21}u'_2\Big) + \hat\sigma_{21}\Big(\frac{\partial u'_2}{\partial x_1} - A_{12}u'_1\Big) + \hat\sigma_{13}\, \frac{\partial u''_1}{\partial x_3} + \hat\sigma_{31}\Big(\frac{\partial u'_3}{\partial x_1} - A'_1 u'_1\Big)$$

$$+ \hat\sigma_{23}\, \frac{\partial u''_2}{\partial x_3} + \hat\sigma_{32}\Big(\frac{\partial u'_3}{\partial x_2} - A'_2 u'_2\Big) + \hat\sigma_{33}\, \frac{\partial u''_3}{\partial x_3}\Big)\, d\omega.$$

If the strains e_{ij} in (1.2) are approximated as

$$e_{11} = \frac{1}{H_1}\Big(\frac{\partial u'{}_1}{\partial \alpha_1} + u'_2 A_{12} + u'_3 A'_1\Big),$$

$$2e_{12} = \frac{1}{H_1}\Big(\frac{\partial u'_2}{\partial \alpha_1} - u'_1 A_{12}\Big) + \frac{1}{H_2}\Big(\frac{\partial u'_1}{\partial \alpha_2} - u'_2 A_{21}\Big), \tag{2.10}$$

$$2e_{31} = 2e_{13} = \frac{1}{H_1}\Big(\frac{\partial u'_3}{\partial \alpha_1} - u'_1 A'_1\Big) + \frac{\partial u''_1}{\partial x_3} \quad (1 \leftrightarrows 2), \qquad e_{33} = \frac{\partial u''_3}{\partial x_3},$$

then

$$E = \int\limits_{\omega} \sigma_{ij} e_{ij} H_1 H_2\, d\omega. \tag{2.11}$$

We assume that the stresses σ_{ij} are related to the strains (2.10) by the equations

$$\sigma_{ij} = a_{ijks} e_{ks}, \qquad i, j, k, s = 1, 2, 3. \tag{2.12}$$

Since the integrand on the right side of equality (2.9) contains derivatives of the product of the polynomials $\hat\sigma'_{\gamma\delta}u'_\gamma$, $\hat\sigma''_{3\delta}u'_3$, and $\hat\sigma'_{i3}u''_i$, boundary conditions (1.4) are replaced by the conditions

$$\Big[a^{\pm}_{\gamma m}u'_\gamma + (1-a^{\pm}_{\gamma m})\hat\sigma'_{\gamma m}\Big]_{x_m=x^{\pm}_m} = \hat\varphi^{\pm}_{\gamma m}, \qquad \Big[a^{\pm}_{3m}u'_3 + (1-a^{\pm}_{3m})\hat\sigma''_{3m}\Big]_{x_m=x^{\pm}_m} = \hat\varphi^{\pm}_{3m},$$

$$\gamma = 1,2,\ m = 1,2; \tag{2.13}$$

$$\Big[a^{\pm}_{i3}u''_i + (1-a^{\pm}_{i3})\hat\sigma'_{i3}\Big]_{x_3=x^{\pm}_3} = \varphi^{\pm}_{i3}, \quad i = 1,2,3, \tag{2.14}$$

where

$$\hat\varphi^{\pm}_{\gamma m} = \sum_{k=0}^{k=N}(\varphi^{\pm}_{\gamma m})^k P_k, \quad \gamma = 1,2, \qquad \hat\varphi^{\pm}_{3m} = \sum_{k=0}^{k=N-1}(\varphi^{\pm}_{3m})^k P_k,$$

$$(\varphi^{\pm}_{im})^k = \frac{1}{2}(1+2k)\int_{-1}^{1}\varphi^{\pm}_{im}P_k\,d\zeta, \qquad i = 1,2,3, \quad m = 1,2.$$

For $N = 1$, conditions (2.13) imply that the vectors of the forces and bending and twisting moments or the corresponding displacements and rotations are specified on the lateral surfaces.

The two-dimensional problem of the Nth approximation is to find the functions σ'_{ij}, σ''_{ij}, u'_i, and u''_i that satisfy Eqs. (2.3)–(2.7) and (2.10)–(2.14).

The functions $\hat\sigma'_{ij}$, $\hat\sigma''_{ij}$, u'_i, and u''_i, whose derivatives with respect to α_i enter (2.5), (2.6), and (2.10) will be called the basic functions and the other functions $\hat\sigma^k_{ij}$ and u^k_i the auxiliary functions. Accordingly, the coefficients of the polynomials $P_k(\zeta)$ in the expansions

$$\hat\sigma'_{11}, \quad \hat\sigma'_{12}, \quad \hat\sigma''_{31} \quad (1 \leftrightarrows 2), \qquad u'_i \quad (i = 1,2,3) \tag{2.15}$$

will be called the basic functions of the variables α_1 and α_2 ($9N + 6$ basic functions) and those in the expansions

$$\sigma'_{31} - \sigma''_{31}, \quad \hat\sigma'_{32} - \hat\sigma''_{32}, \quad \hat\sigma'_{i3}, \quad u''_i - u'_i \quad (i = 1,2,3) \tag{2.16}$$

will be called the auxiliary functions ($3N + 13$ auxiliary functions).

From (2.3), (2.4), and (2.12), it follows that

$$\int_{-1}^{1}(\hat\sigma'_{13} - H_1H_2a_{13ij}e_{ij})P_k\,d\zeta = 0 \quad (1 \leftrightarrows 2), \quad k = 0,1,2,\ldots,N+1,$$

$$\int_{-1}^{1}(\hat\sigma'_{33} - H_1H_2a_{33ij}e_{ij})P_k\,d\zeta = 0, \quad k = 0,1,2,\ldots,N, \tag{2.17}$$

$$\int_{-1}^{1}(\hat\sigma'_{31} - H_2a_{31ij}e_{ij})P_N\,d\zeta = 0 \quad (1 \leftrightarrows 2).$$

Equations (2.14) and (2.17) constitute a closed system of equations for the auxiliary functions (2.16). Provided that

$$u'_i = \frac{\partial u'_i}{\partial\alpha_1} = \frac{\partial u'_i}{\partial\alpha_2} = \varphi^{\pm}_{i3} = 0 \quad (i = 1,2,3) \tag{2.18}$$

any solution of this system satisfies the equalities

$$\int_{-1}^{1}\frac{\partial}{\partial x_3}(\hat\sigma'_{i3}u''_i)\,d\zeta = 0, \qquad \int_{-1}^{1}u''_i\frac{\partial\hat\sigma'_{i3}}{\partial x_3}\,d\zeta = 0,$$

$$\int_{-1}^{1}\hat\sigma'_{i3}\frac{\partial u''_i}{\partial x_3}\,d\zeta = \int_{-1}^{1}a_{ijks}e_{ks}H_1H_2\,d\zeta = 0, \quad e_{11} = e_{12} = 0, \quad 2e_{31} = e_{13} = \frac{\partial u''_i}{\partial x_3} \quad (1 \leftrightarrows 2), \quad e_{13} = \frac{\partial u''_3}{\partial x_3}.$$

Thus, if equalities (2.18) are satisfied, the zero solution of system (2.14), (2.17) is unique and this system is resolvable for the functions (2.16). From the solution of system (2.14), (2.17), the auxiliary functions are expressed in terms of $\varphi_{i3}^{\pm}$ and the coefficients of the polynomials u_i', $\partial u_i'/\partial\alpha_1$, and $\partial u_i'/\partial\alpha_2$ $(i = 1, 2, 3)$. Using these expressions, the equations of the two-dimensional problem can be formulated only for the basic functions (2.15).

From (2.3), (2.4), and (2.12), we obtain

$$\int_{-1}^{1} (\hat{\sigma}_{11}' - H_2 a_{11ij} e_{ij}) P_k \, d\zeta = 0,$$

$$\int_{-1}^{1} (\hat{\sigma}_{12}' - H_1 a_{12ij} e_{ij}) P_k \, d\zeta = 0 \quad (1 \leftrightarrows 2), \quad k = 0, 1, 2, \ldots, N, \tag{2.19}$$

$$\int_{-1}^{1} (\hat{\sigma}_{31}'' - H_2 a_{31ij} e_{ij}) P_k \, d\zeta = 0 \quad (1 \leftrightarrows 2), \quad k = 0, 1, 2, \ldots, N-1.$$

Expressing the auxiliary functions in Eqs. (2.2), (2.6), and (2.19) in terms of the basic functions, we obtain a closed system of equations for the basic functions. This system with boundary conditions (2.13) is the boundary-value problem for $9N + 6$ basic functions (2.15). Given these functions, one can determine the auxiliary functions and, hence, the strains by formulas (2.10) and the stresses by formulas (2.12) at any point of the shell.

3. First-Approximation Equations of an Elastic Layer. To construct two-dimensional equations of an elastic layer in a first approximation, we require that the following conditions be satisfied.

1. The requirement that the stresses should satisfy the equilibrium equations of an infinitely small element is relaxed, namely, the stresses should satisfy the equilibrium equations of an element whose dimensions are infinitely small in the α_1 and α_2 directions and finite in the x_3 direction. Thus, the stresses should satisfy Eqs. (1.5).

2. If the thickness of the layer is small, then, by virtue of the Saint-Venant principle, the conditions at its lateral surfaces (2.13) can be divided into two groups: the conditions that affect the solution over the entire region occupied by the layer (we call these the basic conditions) and the conditions that affect the solution only in the neighborhood of the end faces. In constructing the two-dimensional equations, we require that the boundary-value problem be resolvable for any type of the basic boundary conditions (2.13) and the order of the system of differential equations do not depend on whether stresses, displacements or their linear combination are specified on the faces of the layer.

3. The solution of the two-dimensional equations of the layer should satisfy the energy identity (2.9), which ensures the uniqueness of a certain class of contact problems [6].

For $N = 1$, the layer equations based on the representations for stresses (2.3) and displacements (2.7) satisfy the requirements mentioned above. In the first-approximation equations, the basic functions are the following quantities used in shell theory: the forces, the moments, and the corresponding average displacements and rotations of cross sections of the layer.

In the general case of the stress–strain state, the first-approximation system of differential equations of the elastic layer is a tenth-order system of partial differential equations and solutions of the boundary-value problems can be obtained only using numerical methods. The coefficients of expansions (2.3) and (2.7) are functions of the variables α_1 and α_2. Replacing these functions in the unit square $\{-1 \le \alpha_1 \le 1, \ -1 \le \alpha_2 \le 1\}$ by truncated Legendre polynomials of α_1 and α_2, one can construct the moment finite element [9]. In [10], an iterative algorithm for solving plane elasticity problems based on similar finite elements was proposed and the problem of a cracked plate in tension was solved. A comparison of the numerical and analytical solutions shows that the first-approximation layer equations can effectively be used to solve problems with singularities in the stress state.

For the one-dimensional stress–strain state, the first-approximation system of equations of the elastic layer is a sixth-order system of ordinary differential equations (generally, with variable coefficients). For a circular cylindrical shell and a plane layer, this system becomes a system of differential equations with constant coefficients, for which a general solution can be constructed. We consider these equations for a cylindrical layer. In this case, one should set

$$R_1 = R, \quad A_1 = R, \quad H_1 = R(1 + x_3/R), \quad R_2 = \infty, \quad A_2 = 1, \quad H_2 = 1,$$

$$\hat\sigma_{11} = \sigma_{11}, \quad \hat\sigma_{13} = R(1 + x_3/R)\sigma_{13}, \quad \hat\sigma_{31} = \sigma_{31}, \quad \hat\sigma_{33} = R\sigma_{33}.$$

The forces and moments are calculated by the formulas

$$T_{11} = \int\limits_{-h/2}^{h/2} \sigma_{11}\, dx_3 = \frac{h}{2}\int\limits_{-1}^{1} \sigma_{11}\, d\zeta, \qquad M_{11} = \int\limits_{-h/2}^{h/2} \sigma_{11} x_3\, dx_3 = \frac{h^2}{4}\int\limits_{-1}^{1} \sigma_{11}\zeta\, d\zeta,$$

$$T_{31} = \int\limits_{-h/2}^{h/2} \sigma_{31}\, dx_3 = \frac{h}{2}\int\limits_{-1}^{1} \sigma_{31}\, d\zeta.$$

In the first approximation, the expansions for the stresses are given by

$$\hat\sigma_{11} = \sigma_{11} = \sigma_{11}^0 P_0 + \sigma_{11}^1 P_1, \qquad \hat\sigma_{31} = \sigma_{31} = \sigma_{31}^0 P_0 + \sigma_{31}^1 P_1,$$
$$\hat\sigma_{13} = R\Big(\sigma_{13}^0 + \frac{h}{6R}\,\sigma_{13}^1\Big) P_0 + R\sigma_{13}^1 P_1 + R\sigma_{13}^2 P_2. \tag{3.1}$$

The forces and moments are expressed in terms of the stress expansion coefficients by the formulas

$$T_{11} = h\sigma_{11}^0, \qquad M_{11} = (h^2/6)\sigma_{11}^1, \qquad T_{31} = h\sigma_{31}^0.$$

In this case, the equilibrium equations are written as

$$\frac{\partial\sigma_{11}^0}{R\,\partial\varphi} + \frac{2}{h}\,\sigma_{31}^1 + \frac{\sigma_{31}^0}{R} + q_1^0 = 0,$$
$$\frac{\partial\sigma_{11}^1}{R\,\partial\varphi} + \frac{6}{h}\,\sigma_{13}^2 + \frac{\sigma_{13}^1}{R} + q_1^1 = 0, \qquad \frac{\partial\sigma_{13}^1}{R\,\partial\varphi} + \frac{2}{h}\,\sigma_{33}^1 - \frac{\sigma_{11}^0}{R} + q_3^0 = 0. \tag{3.2}$$

We write Hooke's law relations as

$$\sigma_{11} = \alpha(e_{11} + \gamma e_{33}), \qquad \sigma_{33} = \alpha(e_{33} + \gamma e_{11}), \qquad \sigma_{13} = 2me_{13}, \tag{3.3}$$

where $\alpha = 2/(1-\gamma)$ and $\gamma = \nu/(1-\nu)$ for plane strain and $\gamma = \nu$ for plane stress. In (3.3), the stresses are normalized to the shear modulus μ.

The displacements are approximated by the following Legendre polynomial expansions:

$$u_1' = u_1^0 P_0 + u_1^1 P_1, \qquad u_3' = u_3^0 P_0,$$
$$u_1'' = u_1^0 P_0 + u_1^1 P_1 + u_1^2 P_2 + u_1^3 P_3, \qquad u_3'' = u_3^0 P_0 + u_3^1 P_1 + u_3^2 P_2. \tag{3.4}$$

In accordance with (3.4), the strains are expressed as

$$e_{11} = e_{11}^0 P_0 + e_{11}^1 P_1, \quad e_{33} = e_{33}^0 P_0 + e_{33}^1 P_1, \quad e_{13} = e_{13}^0 P_0 + e_{13}^1 P_1 + e_{13}^2 P_2, \tag{3.5}$$

where

$$e_{11}^0 = \frac{1}{R}\Big(\frac{\partial u_1^0}{\partial\varphi} + u_3^0\Big), \qquad e_{11}^1 = \frac{1}{R}\,\frac{\partial u_1^1}{\partial\varphi},$$

$$2e_{13}^0 = \frac{1}{R}\Big(\frac{\partial u_3^0}{\partial\varphi} - u_1^0 + \frac{2R}{h}\,(u_1^1 + u_1^3)\Big), \qquad 2e_{13}^1 = \frac{1}{R}\,u_1^3 + \frac{6}{h}\,u_1^2,$$

$$2e_{13}^2 = \frac{10}{h}\,u_1^3, \qquad e_{33}^0 = \frac{2}{h}\,u_3^1, \qquad e_{33}^1 = \frac{6}{h}\,u_3^2.$$

From (3.1), (3.4), and (3.5), we obtain

$$\sigma_{11}^0 = \alpha\Big[\frac{1}{R}\Big(\frac{\partial u_1^0}{\partial\varphi} + u_3^0\Big) + \gamma\,\frac{2}{h}\,u_3^1\Big], \qquad \sigma_{11}^1 = \alpha\Big(\frac{1}{R}\,\frac{\partial u_1^1}{\partial\varphi} + \gamma\,\frac{6}{h}\,u_3^2\Big),$$

$$\sigma_{33}^0 = \alpha\Big[\gamma\,\frac{2}{h}\,u_3^1 + \gamma\,\frac{1}{R}\Big(\frac{\partial u_1^0}{\partial\varphi} + u_3^0\Big)\Big], \qquad \sigma_{33}^1 = \alpha\Big(\frac{6}{h}\,u_3^2 + \gamma\,\frac{1}{R}\,\frac{\partial u_1^1}{\partial\varphi}\Big), \tag{3.6}$$

$$\sigma_{13}^0 = m\,\frac{1}{R}\Big(\frac{\partial u_3^0}{\partial\varphi} - u_1^0 + \frac{2R}{h}\,(u_1^1 + u_1^3)\Big), \quad \sigma_{13}^1 = m\Big(\frac{u_1^3}{R} + \frac{6}{h}\,u_1^2\Big), \quad \sigma_{13}^2 = m\,\frac{10}{h}\,u_1^3.$$

Ten equations (3.2) and (3.6) and four conditions on the layer faces

$$\begin{gathered}
\alpha(u_1^0 + u_1^1 + u_1^2 + u_1^3) + (1-\alpha)(\sigma_{13}^0 + (1 + 1/(6R))\sigma_{13}^1 + \sigma_{13}^2) = \varphi_{13}^+(\xi), \\
\alpha(u_1^0 - u_1^1 + u_1^2 - u_1^3) + (1-\alpha)(\sigma_{13}^0 - (1 + 1/(6R))\sigma_{13}^1 + \sigma_{13}^2) = \varphi_{13}^-(\xi), \\
\alpha(u_3^0 + u_3^1 + u_3^2) + (1-\alpha)R(\sigma_{33}^0 + \sigma_{33}^1) = \varphi_{33}^+(\xi), \\
\alpha(u_3^0 - u_3^1 + u_3^2) + (1-\alpha)R(\sigma_{33}^0 - \sigma_{33}^1) = \varphi_{33}^-(\xi)
\end{gathered} \tag{3.7}$$

form a closed system of equations for the six basic functions

$$u_1^0, \quad u_1^1, \quad u_3^0, \quad \sigma_{11}^0, \quad \sigma_{11}^1, \quad \sigma_{13}^0$$

and the eight auxiliary functions

$$u_1^2, \quad u_1^3, \quad u_3^1, \quad u_3^2, \quad \sigma_{13}^2, \quad \sigma_{11}^3, \quad \sigma_{33}^0, \quad \sigma_{33}^1.$$

We introduce the dimensionless variables

$$\bar\sigma_{ij} = \frac{\sigma_{ij}}{\sigma_0}, \quad \bar\varepsilon_{ij} = \frac{\varepsilon_{ij}}{\varepsilon_0}, \quad \varepsilon_0 = \frac{\sigma_0}{\mu}, \quad \bar u_i = \frac{2u_i}{h\varepsilon_0},$$

$$\xi = \frac{x_1}{L_0}, \quad \zeta = \frac{2x_3}{h}, \quad \eta = \frac{h}{2R}, \quad \bar q_i^0 = \frac{q_i^0 h}{2\sigma_0}$$

(σ_0 and L_0 are characteristic quantities having the dimensions of stress and length, respectively, and ε_0 is the characteristic strain). Below, the bar above the dimensionless quantities is omitted.

In the dimensionless variables, the approximations of stresses (3.1) and displacements (3.4) are written as

$$\begin{gathered}
\sigma_{11}' = t_{11} + m_{11}P_1, \qquad \sigma_{13}' = t_{13} + m_{13}P_1 + r_{13}P_2, \\
\sigma_{31}' = t_{31}, \qquad \sigma_{33}' = t_{33} + m_{33}P_1, \\
u_1' = u_0 + u_1P_1, \quad u_3' = v_0, \quad u_1'' = u_0 + u_1P_1 + u_2P_2 + u_3P_3, \quad u_3'' = v_0 + v_1P_1 + v_2P_2,
\end{gathered} \tag{3.8}$$

where

$$t_{11} = \frac{T_{11}}{h\sigma_0}, \qquad t_{31} = t_{13} = \frac{T_{21}}{h\sigma_0}, \qquad m_{11} = \frac{6M_{11}}{h^2\sigma_0},$$

$$u_0 = \frac{1}{h\varepsilon_0}\int_{-h/2}^{h/2} \frac{u_1}{h}\,dx_3, \qquad u_1 = \frac{6}{h^2\varepsilon_0}\int_{-h/2}^{h/2} \frac{u_1}{h}\,x_3\,dx_3, \qquad v_0 = \frac{1}{h\varepsilon_0}\int_{-h/2}^{h/2} \frac{u_3}{h}\,dx_3.$$

In the dimensionless variables, the system of differential equations for the expansion coefficients (3.8) becomes

$$\begin{gathered}
\eta(t_{11}' + t_{13}) + (\sigma_{13}^+ - \sigma_{13}^-)/2 + q_1^0 = 0, \qquad \eta(t_{13}' + t_{11}) + (\sigma_{33}^+ - \sigma_{33}^-)/2 + q_3^0 = 0, \\
\eta m_{11}' - 3t_{13} + 3(\sigma_{13}^+ + \sigma_{13}^-)/2 + q_1^1 = 0, \\
t_{11} = \alpha(\eta(u_0' + v_0) + \gamma v_1), \qquad t_{33} = \alpha(\gamma\eta(u_0' + v_0) + v_1), \\
m_{11} = \alpha(\eta u_1' + 3\gamma_1 v_2), \qquad m_{33} = \alpha(\gamma\eta u_1' + 3v_2), \\
t_{13} = (\eta(v_0' - u_0) + u_1 + u_3), \qquad m_{13} = 3(u_2 + v_1), \qquad r_{12} = 5mu_3.
\end{gathered} \tag{3.9}$$

From Eqs. (3.7), the auxiliary quantities can be expressed in terms of the basic quantities and functions specified on the layer surface. Substitution of these expressions into Eqs. (3.9) yields a sixth-order system of differential equations for the basic quantities, whose order does not depend on the boundary conditions on the layer faces.

Introducing the vector

$$\boldsymbol{Z} = [u_0,\ u_1,\ v_0,\ t_{11},\ m_{11},\ t_{13}]^{\mathrm{t}},$$

we write the first-approximation system of equations of the cylindrical layer as

$$\boldsymbol{Z}' = H\boldsymbol{Z} + \boldsymbol{F}, \tag{3.10}$$

where H is a 6×6 matrix and $\boldsymbol{F}$ is a six-component vector.

For $\xi = \xi_0$ and $\xi = \xi_1$, system (3.10) is subject to boundary conditions of the form

$$A\boldsymbol{X} + B\boldsymbol{Y} = \boldsymbol{C},$$

where

$$\boldsymbol{X} = \begin{Vmatrix} u_0 \\ u_1 \\ v_0 \end{Vmatrix}, \qquad \boldsymbol{Y} = \begin{Vmatrix} t_{11} \\ m_{11} \\ t_{12} \end{Vmatrix},$$

A and B are specified 3×3 matrices and $\boldsymbol{C}$ is a specified three-component vector.

Replacing $1/R$ in Eqs. (3.7) and (3.9) by $\eta = h/L_0$, we obtain the equations of an elastic plane layer [7]. The general solutions of the equations of an elastic plane layer for various conditions on the layer faces are given in [7]. The equations of an elastic plane layer and their general solutions given in [7] were used to solve a number of contact problems with mixed boundary conditions on the layer faces [10–13].

4. Equations of Laminated Shells. For a shell composed of several elastic layers, one can write the equations given in Secs. 1–3 for each layer. The resulting system should be supplemented by matching conditions for the stresses and displacements at the interfaces. These conditions are formulated in terms of the following truncated Legendre polynomials:

$$\hat{\sigma}'_{13}, \quad \hat{\sigma}'_{23}, \quad \hat{\sigma}''_{33}$$

for the stresses and

$$u''_1, \quad u''_2, \quad u''_3$$

for the displacements.

The stress–strain state in laminated shells can be determined employing various numerical algorithms. Using matching conditions for the stresses and displacements at the interfaces between the layers, one can construct a system of differential equations for a laminated shell. However, this algorithm is ineffective for a large number of layers in the package. In this case, the problem can be solved through the use of an iterative algorithm [10].

Conclusions. The equations derived in the present paper admit formulation of mixed conditions on the shell faces, which can be used in solving contact problems with unknown interface between regions with different boundary conditions. Moreover, these equations take into account finite shear rigidity. The approach described in the paper can be used to construct shell equations in other curvilinear coordinate systems [14–16]. The algorithm for constructing the shell equations remains unaltered if the coefficients a_{ijks} in (2.10) are functions of the variables α_1, α_2, and x_3. Therefore, the approach proposed can be used to construct equations for inhomogeneous anisotropic shells [17] and for nonlinear constitutive relations.

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